THE UNIFORM MIXTURE OF GENERALIZED ARC-SINE DISTRIBUTIONS

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Abstract

A single, tractable, special case of the problem of continuous mixtures of beta distributions over their parameters is considered. This is the uniform mixture of generalized arc-sine distributions which, curiously, turns out to be linked by transformation to the Cauchy distribution.

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Let Beta(a, b) stand for the usual beta distribution on $[0, 1]$ with parameters $a, b > 0$. Mixtures of beta distributions are commonly found in the statistical literature, but arise almost always in the form of discrete mixtures over a and/or b or else, very occasionally, as mixtures over an additional scale parameter (Bertin *et al.*, 1997). Continuous mixtures over a and b are conspicuous by their absence. This is unsurprising on tractability grounds, given the need to integrate over functions of a and b including a beta function.

In this note, it is observed that one very special case of a continuous mixture of beta distributions is simple and tractable. This special case is the distribution of U where $U|\Theta = \theta \sim$ Beta$(\theta, 1-\theta)$, $0 < \theta < 1$, and $\Theta \sim U(0, 1)$, the continuous uniform distribution on $(0, 1)$. The Beta$(\theta, 1-\theta)$ distribution is also known as the generalized arc-sine distribution. Its density is

$$f(u|\theta) = \frac{u^{\theta-1}(1-u)^{-\theta}}{B(\theta, 1-\theta)} = \frac{\sin(\pi\theta)}{\pi u}\left(\frac{u}{1-u}\right)^{\theta}, \qquad 0 < u < 1,$$

so that the density of its uniform mixture is

$$\begin{aligned} f(u) &= \frac{1}{\pi u}\int_0^1 \sin(\pi\theta)\exp\{\theta\log(u/(1-u))\}d\theta \\ &= \frac{1}{u(1-u)}\frac{1}{\left\{\pi^2 + \log^2\left(\frac{u}{1-u}\right)\right\}}, \qquad 0 < u < 1. \end{aligned} \tag{1}$$

Five representative densities of the generalised arc-sine family ($\theta = 0.1(0.2)0.9$) are plotted in Fig. 1 along with density (1) (the dashed line). (All densities tend to ∞ as $u \to 0, 1$, although u has to be very close to 0 or 1 for this to be apparent for θ close to 1 or 0, respectively.) Note that the uniform mixture density shares the property of being U-shaped with all individual generalized arc-sine distributions.

* * * Fig. 1 about here * * *

The uniform "averaged generalized arc-sine density" (1) can be immediately recognised as the distribution of $U = e^{\pi X}/(1 + e^{\pi X})$ where $X \sim$ standard Cauchy. As such, it arises from the logistic transformation subcase of the transformation approach of Johnson (1949) applied to Cauchy rather than normal random variables. This, in turn, means that distribution (1) is also one of a tractable family of distributions on [0,1] that form alternatives to the beta distribution; Johnson and Tadikamalla (1982) provide, through similar transformation of the logistic, another attractive family.

The uniform mixture distribution can be contrasted with the "median generalized arc-sine density", that of Beta$(1/2, 1/2)$, the ordinary arc-sine density, which is the symmetric member of the set of generalized arc-sine densities shown in Fig. 1. This is the distribution of $V = (1/2)(1+X/\sqrt{1+X^2})$, a simple consequence of the relationship of Cacoullos (1965). Note that the latter has more probability mass than the former in the interior of [0,1] and less probability mass very close to the boundaries. This is a consequence of the relative spreads of the respective transformations.

The author's motivation for this study was his interest in perhaps replacing a beta kernel estimator of a density on $[0, 1]$, due to Chen (2000), which is of the form

$$n^{-1} \sum_{i=1}^{n} K\left(Y_i; (y/w) + 1, ((1 - y)/w) + 1\right)$$

by its natural converse

$$n^{-1} \sum_{i=1}^{n} K\left(y; (Y_i/w) + 1, ((1 - Y_i)/w) + 1\right).$$

Here, $Y_1, ..., Y_n$ are the data, y is the point at which estimation takes place, w is a smoothing parameter and $K(z; a, b)$ is the beta density in z with parameters a and b. The expectation of the latter is a continuous mixture of beta distributions over a distribution for their parameters.

However, it is not clear that any continuous mixtures of beta distributions over a and b other than the simple one discussed in this note afford such explicit solution. This and the intriguing link between the uniform mixture of generalized arc-sine distributions and the Cauchy distribution are what make the note stand on its own.

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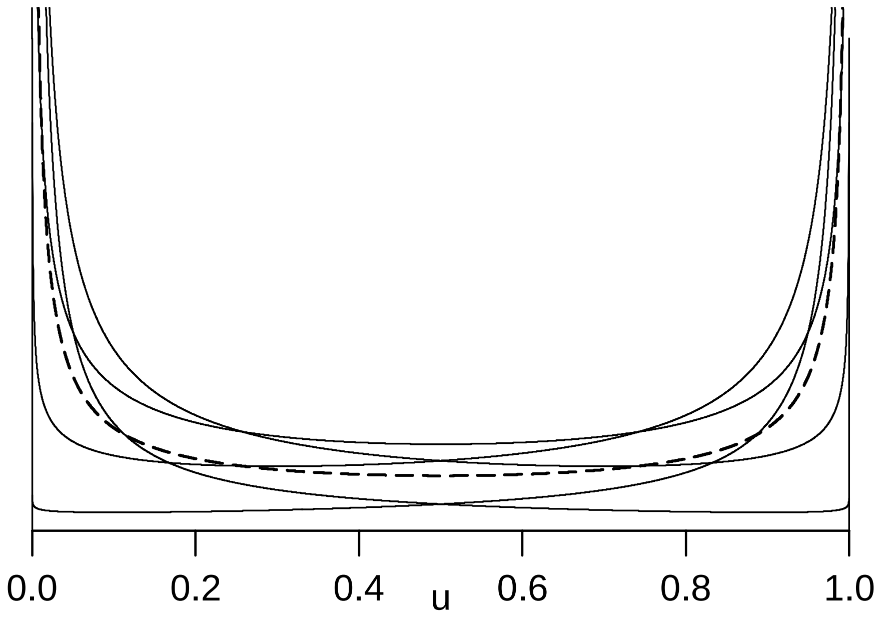
FIGURE LEGEND

Figure 1 *Five generalized arc-sine densities (solid lines) and the uniform mixture density (1) (dashed line). To identify individual generalized arc-sine densities, note that at* $u = 0.4$*, the densities correspond to* $\theta = 0.9, 0.1, 0.7, 0.3$ *and* 0.5 *in increasing values of* $f(0.4)$*.*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